A Case for Strongly Polynomial Time Sub-Polyhedral Scheduling Using Two-Variable-Per-Inequality Polyhedra

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Outline

- Unscalability of Polyhedral Scheduling
- Sub-Polyhedral Varieties and Summary (TVPI and UTVPI)
- Sub-Polyhedral Scheduling
- Under-Approximation Algorithms: TVPI
  - The problem of finding Under-Approximation
  - Convexity, TVPI-UA and per-constraint methods
  - Median method and LP based methods
  - Theoretical comparison
- Experimental Results
- Conclusions and Future Work

Unscalability of Polyhedral Scheduling: A Motivation

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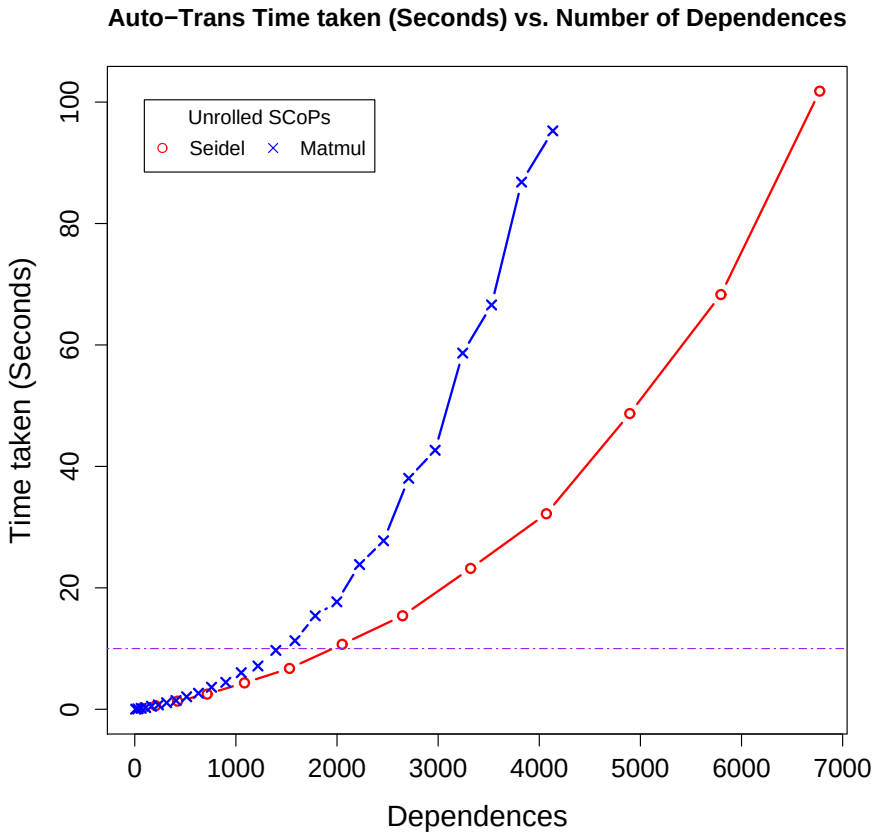

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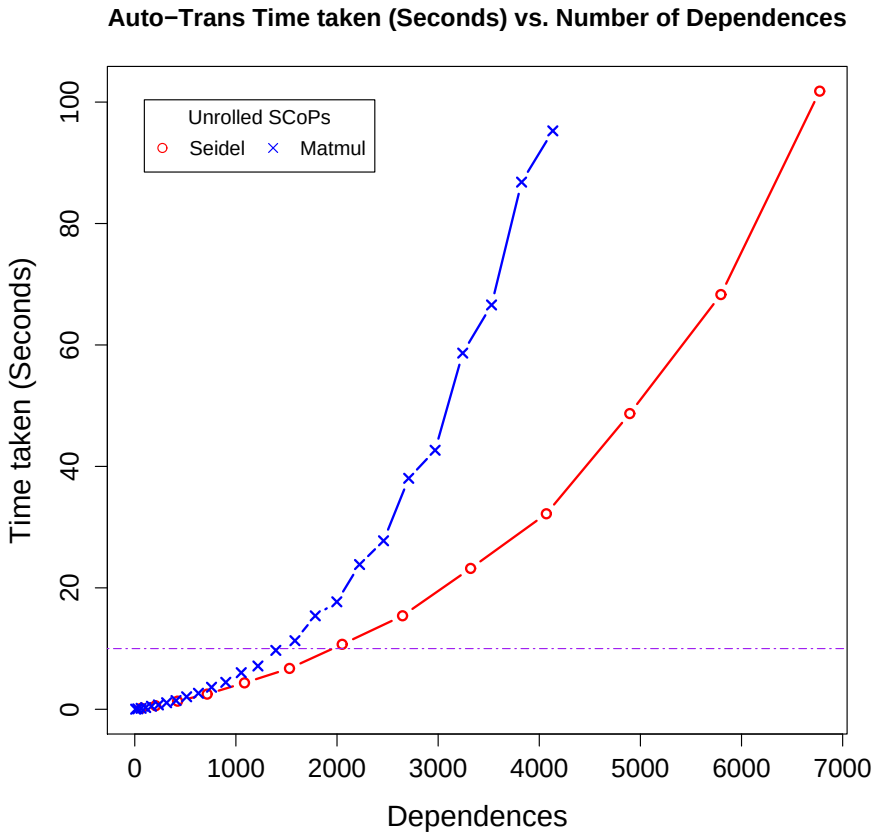

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  ($|V|$: #statements)
- $>$ 10s compilation limit

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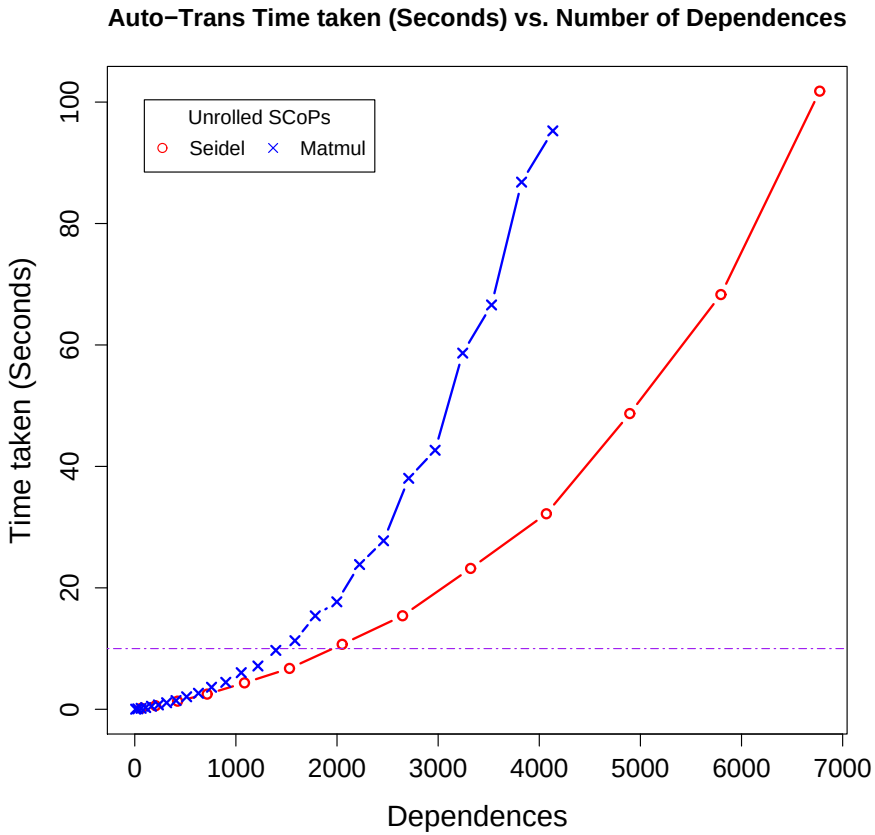

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Our solution: Algorithms using approximations of polyhedra.

Sub-Polyhedra

Sub-Polyhedral Varieties

Theme: Trade precision for cost.

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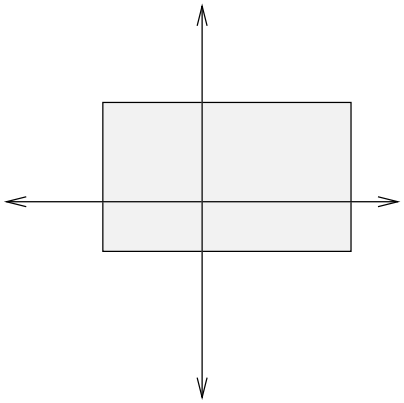
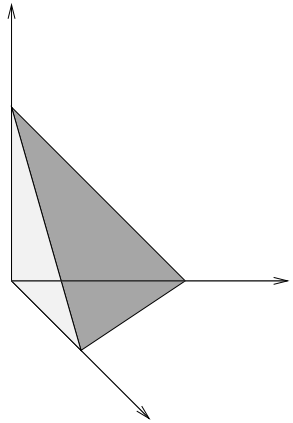


Interval

$$a \leq x_i \leq b$$



Convex Polyhedra

$$\sum a_i x_i \leq c$$

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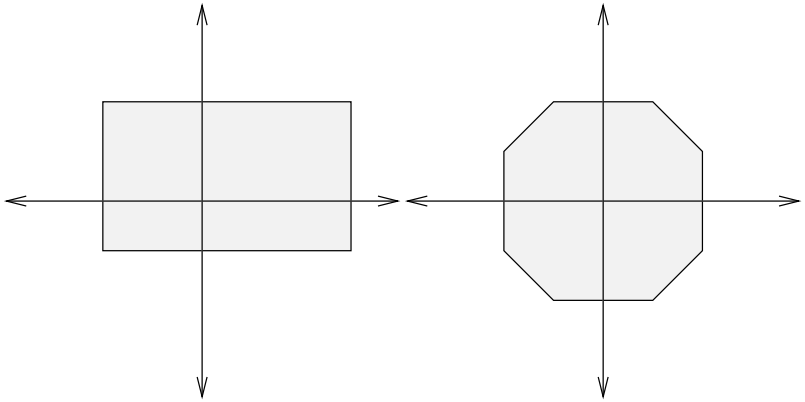

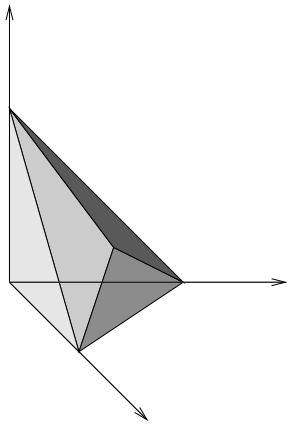

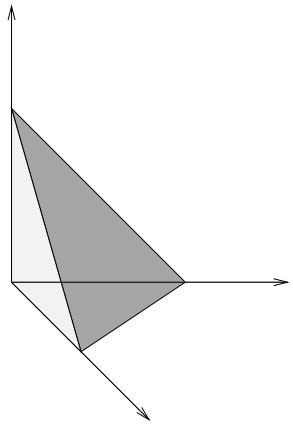

Interval	Octagon (UTVPI)	TVPI	Convex Polyhedra
	Unit Two Variable Per Inequality	Two Variable Per Inequality	
$a \leq x_i \leq b$	$\pm x_i \pm x_j \leq c$	$ax_i + bx_j \leq c$	$\sum a_i x_i \leq c$

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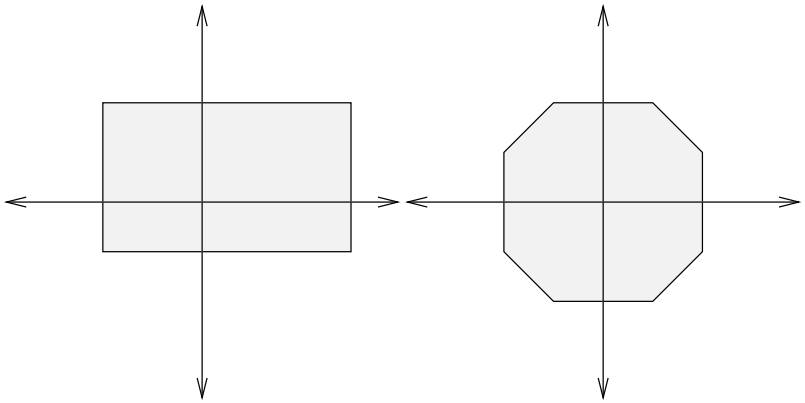

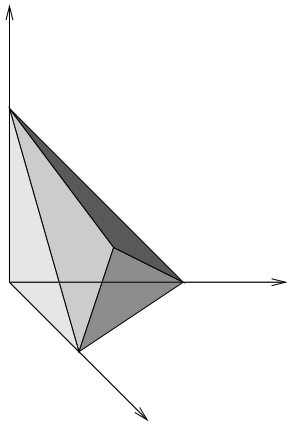

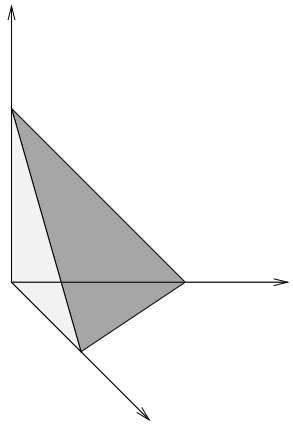

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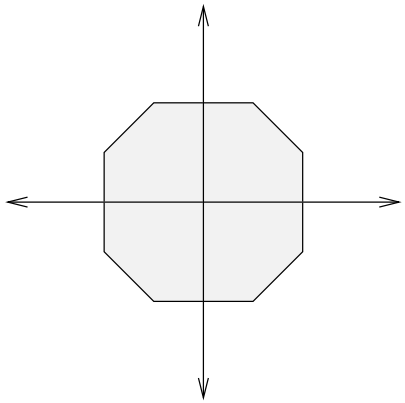
$$\xrightarrow[\text{Precision}]{}$$

$$\text{Intervals} \subset \text{Octagons (UTVPI)} \subset \text{TVPI} \subset \text{Conv.Poly}$$

$$\xrightarrow[\text{Cost}]{}$$

Sub-Polyhedra Summary: TVPI and UTVPI



UTVPI (Octagon)

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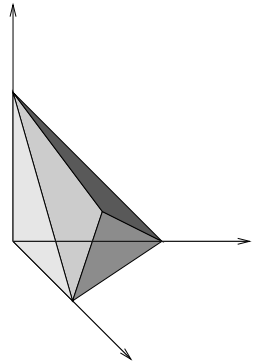

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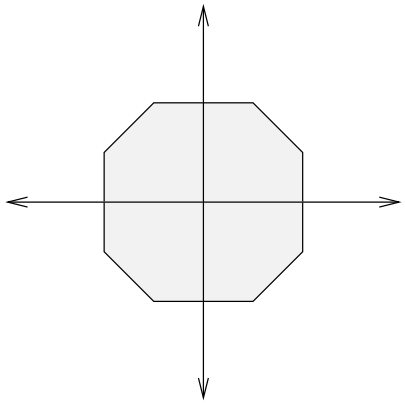

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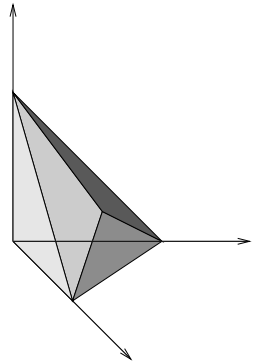

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Feasibility	$\Theta(mn)$ (Bellman-Ford)	$\Theta(mn^2 \log m)$ Hochbaum-Naor-94 (Fourier-Motzkin special case)

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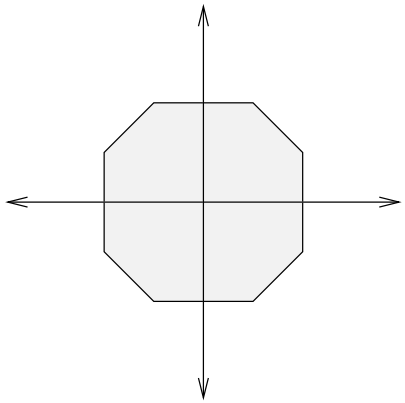

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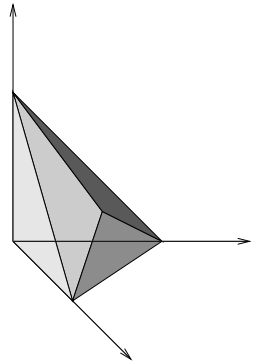

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Optimization	$\Theta(mn)$ (Bellman-Ford)	$\Theta(m^3 n^2 \log A)$ Wayne-99 (Min-cost flow)

Better worst case time complexities than general (convex) polyhedra.

Sub-Polyhedral Scheduling

Schedule Polyhedron Approximation: Motivation

Reminder: Feautrier's scheduler:

Input: Dependence polyhedron per e: $\mathcal{D}_e(\lambda, \mathcal{I}, \mathcal{N})$

- Apply Farkas lemma $\Longrightarrow$ Farkas polyhedron $\mathcal{P}_e(\mu, \lambda)$ on edge e.
- Construct overall system $\mathcal{P} = \bigcap_{e \in E} \mathcal{P}_e$.
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- Legal because $\mathcal{P}$ describes the set of solution points.
  (UA $\Longrightarrow$ we will be throwing away some of them.)
- Works for any *Feautrier-family* of affine transformations.
  (Ex: Forward Communications Only transformation in PLuTo.)
- Each $\mathcal{P}_e$ is *very* small $\Longrightarrow$ UA can be done per dependence edge.

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New constraint system: $\mathcal{P}_a = \bigcap_{e \in E} \mathrm{UA}(\mathcal{P}_e)$

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Scheduling algorithm using TVPI approximation of constraints

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- worstcase strongly polynomial time
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Question: How to find $\mathcal{P}_a$?

TVPI Under-Approximation Algorithms

The Problem of Finding Under-Approximation

Input: General convex polyhedron $\mathcal{P}$ in constraint form.
Goal: Compute $\mathcal{P}_a = \mathrm{UA}_{\mathrm{TVPI}}(\mathcal{P}) \subseteq \mathcal{P}$

- A guarantee that $\mathcal{P}_a \neq \phi$ for non-trivial cases.
- Simple algorithm:
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Our solution strategy : Use dual transformation

$$\mathcal{P} \rightarrow \mathcal{P}^* \rightsquigarrow \mathrm{OA}(\mathcal{P}^*) \rightarrow \mathrm{UA}(\mathcal{P})$$

- Only a *re-interpretation* of $\mathcal{P}$'s constraints in dual space.
- An approximation scheme $\Rightarrow$ family of approximation algorithms.

Convexity Basics and TVPI Approximation

Primal Space

Dual Space

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UA of system $\mathcal{P}$ (constraint form) $\mathcal{P} \supseteq \mathcal{P}_a$	$\equiv$	OA of system $\mathcal{K}$ (generator form) $\mathcal{K} \subseteq \mathcal{K}_a$

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$$\mathcal{P} = \{\mathbf{x} | \mathbf{a}_1\mathbf{x} \geq b_1; \ldots; \mathbf{a}_j\mathbf{x} \geq b_j; \ldots; \mathbf{a}_m\mathbf{x} \geq b_m;\}$$

finding

$$\mathcal{P}_a = \{\mathbf{x} | \mathbf{a}_1\mathbf{x} \geq b_1; \ldots; \mathrm{UA}(\mathbf{a}_j\mathbf{x} \geq b_j); \ldots; \mathbf{a}_m\mathbf{x} \geq b_m;\}$$

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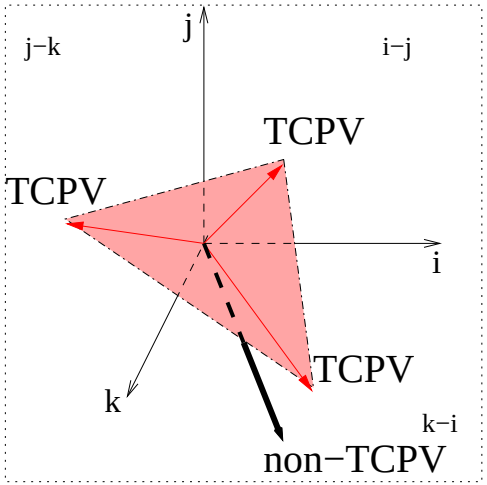

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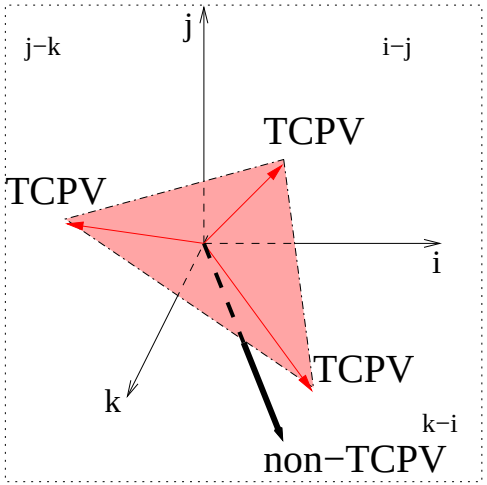

Solution: The decomposition is a TCPV-OA of the vector!

Median Method for Under-Approximation

The Median Method of Approximation

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$$\mathcal{K}_a = \text{cone}\left\{ \begin{pmatrix} \mathbf{a}_1^T \\ b_1 \end{pmatrix}, \ldots, \begin{pmatrix} \frac{1}{2}a_{j,1} & \frac{1}{2}a_{j,1} & 0 \\ \frac{1}{2}a_{j,2} & 0 & \frac{1}{2}a_{j,2} \\ 0 & \frac{1}{2}a_{j,3} & \frac{1}{2}a_{j,3} \\ \frac{b_j}{3} & \frac{b_j}{3} & \frac{b_j}{3} \end{pmatrix}, \ldots, \begin{pmatrix} \mathbf{a}_m^T \\ b_m \end{pmatrix}, \begin{pmatrix} \mathbf{0} \\ 1 \end{pmatrix} \right\}$$

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$\Longleftrightarrow$ Use dual of the above strategy for each non-TVPI constraint in $\mathcal{P}$!

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$\Longleftrightarrow$ Use dual of the above strategy for each non-TVPI constraint in $\mathcal{P}$!

$$\mathcal{P}_a = \left\{ \mathbf{x} | \ldots; \frac{1}{2} \begin{pmatrix} a_{j,1} & a_{j,2} & 0 \\ a_{j,1} & 0 & a_{j,3} \\ 0 & a_{j,2} & a_{j,3} \end{pmatrix} \mathbf{x} \geq \frac{1}{3} \begin{pmatrix} b_j \\ b_j \\ b_j \end{pmatrix}; \ldots; \right\}$$

LP based Method for Under-Approximation

LP based Method

Main idea: Use homogenizing dimension $\Longrightarrow$ an LP problem.
(Instead of $\frac{b_j}{3} + \frac{b_j}{3} + \frac{b_j}{3} = b_j$, we use new variables $\mathbf{p}$.)

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- $\sum \mathbf{p}^j = b_j$ is *context constraint.*
- $\mathbf{p}$-vector values can found by LP problem.
- Results in proper UA.

A Comparison of UA Algorithms

Comparison of Algorithms

	Median	LP	
Variants	✗	✔	One-shot, Independent

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Simple to implement	✔	✔	

Empirical Evaluation

Empirical Framework

Framework:

- Experiments done in PLuTo[1].
- PolyBench[2] benchmarks
- Code in C/C++ (uses PIP/PolyLib/FMLib)

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Polyhedral characteristics:

- Most dependence-polyhedral constraints are TVPI.
- A TVPI constraint is *always* a UTVPI constraint.
- Low sparsity $\hat{s} \simeq 4$ & independent of n (#dimensions).
- Need for duplication elimination (even syntactic).

Testing:

- Feasibility: PIP
- Schedule: using $\mathrm{UA}(\mathcal{P})$, usual cost function and PIP

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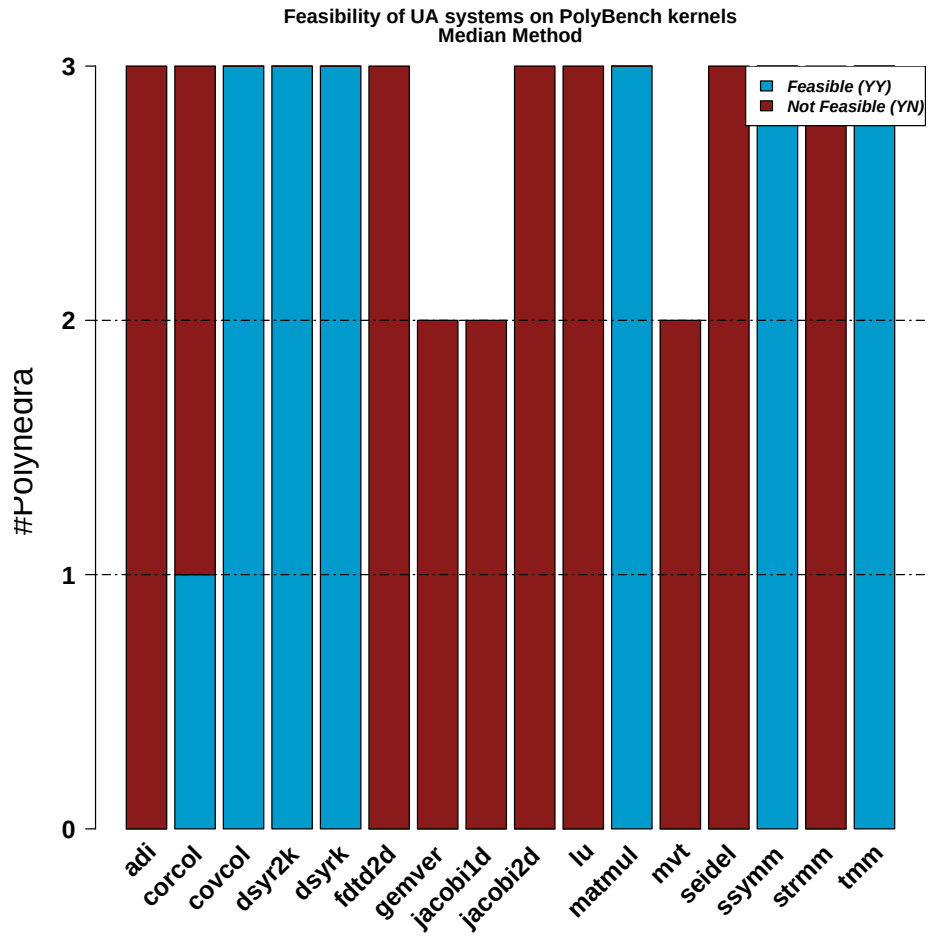
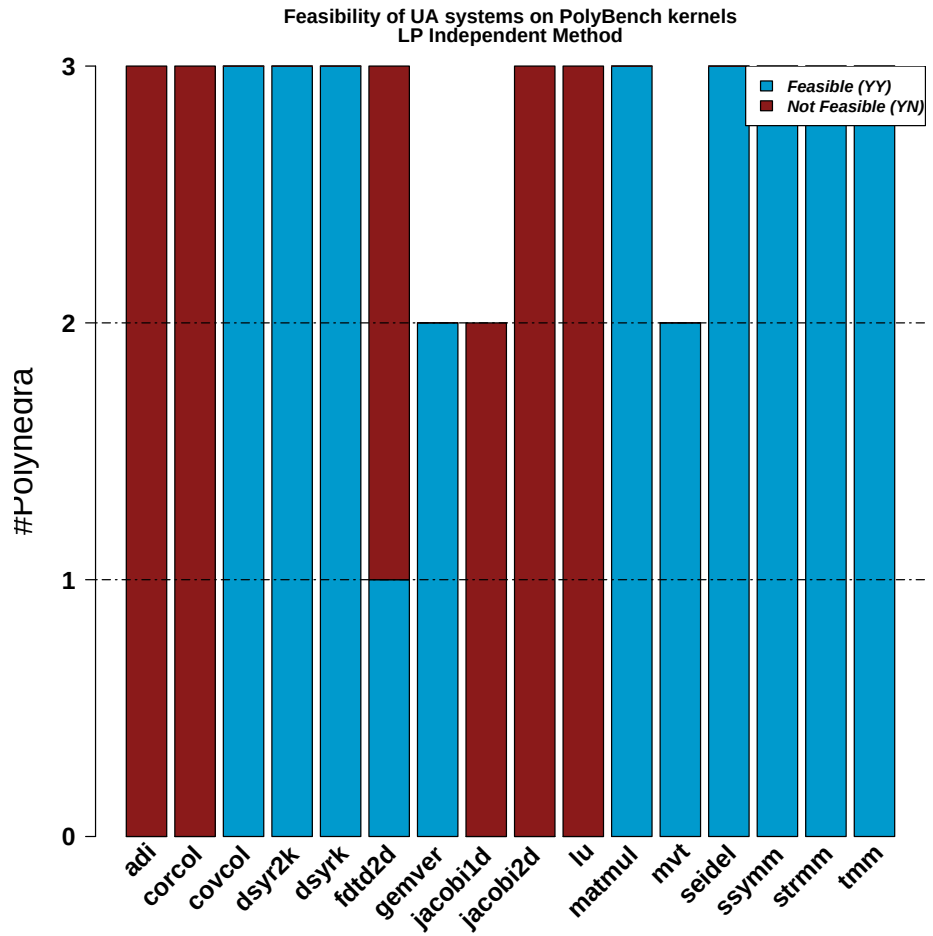
Feasibility of $\mathrm{UA}(\mathcal{P})$: Median Method



- Total 45 Polyhedra: 19 Feasible vs. 26 Not Feasible UAs
- 6 out of 16 PolyBench benchmarks

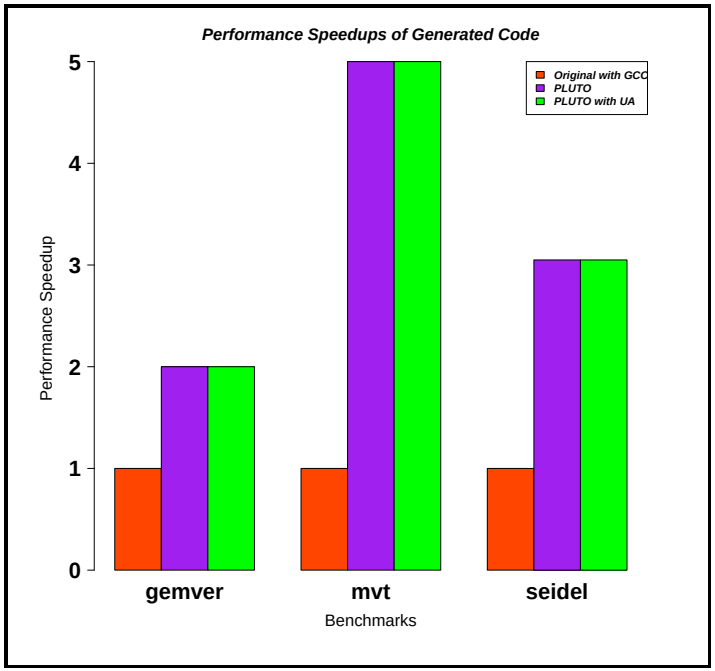
Feasibility of $\mathrm{UA}(\mathcal{P})$: LP-Independent Method



- Total 45 Polyhedra: 29 Feasible vs. 16 Not Feasible UAs
- 10 out of 16 PolyBench benchmarks

Generated Code Comparison (Par)

Details: Schedule found using $\mathrm{UA}(\mathcal{P})$, usual cost function and PIP.



Note: Implementation in progress!

Wrapping up

Summary and Contributions

Summary:

- There exists a scalability problem in polyhedral scheduling.
- Sub-Polyhedra (TVPI/UTVPI) offer better guarantees
  $\Longrightarrow$ scalability in scheduling (precision vs. cost tradeoff)

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Contributions: Sub-Polyhedral (TVPI/UTVPI) UA algorithms

- Median method and LP based methods
  - (-) increase in system size
    $(m \times n \rightsquigarrow \hat{s}^2 m \times n)$
  - (+) ~~vertex method~~ ~~quadratic formulation~~
    Linear formulation (polynomial time)
  - (+) $\mathrm{UA}(\mathcal{P}) \neq \phi$ for considerable (29/45) cases
  - (+) generate code using UA systems
    $T_{exec}(\mathcal{P}) \cong T_{exec}(\mathrm{UA}(\mathcal{P}))$
- Experimental evidence

Challenges and Future Work (Beaucoup!)

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Future work:

- UTVPI approximation $\Longrightarrow$ Bellman-Ford polyhedra
  $\Longrightarrow \Theta(mn)$ polyhedral scheduling: our estimate of lower bound.
- Per dependence implementation.
- Code generation for all feasible cases.

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Take home message: We think that sub-polyhedral scheduling

- is possible using (U)TVPI systems
- will help beat the scalability challenge.

Merci et Questions

Merci Beaucoup!

- Paul Feautrier
- Cédric Bastoul, Armin Größlinger
- Reviewers (IMPACT 2011, 2012)

"Truth is much too complicated to allow anything but approximations" – John von Neumann

Questions?

Backup

Homogenization-Polarity Framework

$$P^{\mathcal{H}} \xrightarrow{\text{homog}} C^{\mathcal{H}} \xrightarrow{\text{polar}} K^{\mathcal{V}} \xrightarrow{\text{OA}} K_a^{\mathcal{V}} \xrightarrow{\text{depolar}} C_a^{\mathcal{H}} \xrightarrow{\text{dehomog}} P_a^{\mathcal{H}}$$

- [Step 0] $P^{\mathcal{H}}$: Input P, the Polyhedron to be under-approximated.
- [Step 1] $P^{\mathcal{H}} \xrightarrow{\text{homog}} C^{\mathcal{H}}$: $C = \text{Homog}(P)$.
- [Step 2] $C^{\mathcal{H}} \xrightarrow{\text{polar}} K^{\mathcal{V}}$: $K = C^*$ (the Polar of C).
- [Step 3] $K^{\mathcal{V}} \xrightarrow{\text{OA}} K_a^{\mathcal{V}}$: $K \subseteq K_a$.
- [Step 4] $K_a^{\mathcal{V}} \xrightarrow{\text{depolar}} C_a^{\mathcal{H}}$:
- [Step 5] $C_a^{\mathcal{H}} \xrightarrow{\text{dehomog}} P_a^{\mathcal{H}}$:

Approximation Scheme

Homogenization-polarity framework: $P \supseteq P_a \iff K \subseteq K_a$

$$K = \text{cone}\left\{\begin{pmatrix} \mathbf{a}_1^T \\ b_1 \end{pmatrix}, \ldots, \begin{pmatrix} \mathbf{a}_j^T \\ b_j \end{pmatrix}, \ldots, \begin{pmatrix} \mathbf{a}_m^T \\ b_m \end{pmatrix}, \begin{pmatrix} \mathbf{0} \\ 1 \end{pmatrix}\right\}$$

$$K \in \text{cone}\,(K_a)$$

$$K^j = \begin{pmatrix} a_{j,1} \\ a_{j,2} \\ a_{j,3} \\ b_j \end{pmatrix} \in \text{cone} \begin{pmatrix} t_1^{j,1} & t_2^{j,1} & 0 \\ t_1^{j,2} & 0 & t_3^{j,2} \\ 0 & t_2^{j,3} & t_3^{j,3} \\ p_1^j & p_2^j & p_3^j \end{pmatrix} = \text{cone}(T^j)$$

Context constraints:

$$T^j = \{t_1^{j,1} + t_2^{j,1} = a_{j,1};\ t_1^{j,2} + t_3^{j,2} = a_{j,2};\ t_2^{j,3} + t_3^{j,3} = a_{j,3};\ p_1^j + p_2^j + p_3^j = b_j\}$$

Polyhedral Characteristics: FPH

	SCoP			FPH						
Bench	#L	#S	#D	P_{FPH}	n	$\hat{m}$	$\hat{m}_T$	$\| \hat{\mathcal{S}}_{m_k} \|$	$\hat{m}_E$	$\hat{m}_U$
adi	12	4	54	3	21	200	65	5	1844	614
corcol	12	6	14	3	23	22	13	5	130	77
covcol	13	7	26	3	25	18	14	4	29	55
dsyr2k	3	1	3	3	8	8	6	3	11	18
dsyrk	3	1	3	3	8	8	7	3	11	18
fdtd-2d	11	4	48	3	21	96	35	6	1010	367
gemver	7	4	13	2	15	21	15	4	48	47
jacobi-1d	4	2	10	2	11	32	14	5	232	104
jacobi-2d	6	2	14	3	13	65	15	7	1135	212
lu	5	2	10	3	11	35	17	5	232	106
matmul	3	1	3	3	10	9	7	4	20	24
mvt	4	2	11	2	10	20	12	3	46	52
seidel	3	1	27	3	9	33	15	5	168	179
ssymm	8	3	15	3	15	15	10	3	22	36

Polyhedral Characteristics: DS/DIR

SCoP				DS/DIR				
Bench	#L	#S	#D	P_{DS}	P_{DIR}	n	$\hat{m}$	$\hat{m}_T$
adi	12	4	54	90	564	10	20	19
corcol	12	6	14	38	194	10	17	16
covcol	13	7	26	41	228	16	25	24
dsyr2k	3	1	3	9	18	9	18	17
dsyrk	3	1	3	9	18	9	18	17
fdtd-2d	11	4	48	39	168	11	22	21
gemver	7	4	13	29	161	7	13	12
jacobi-1d	4	2	10	16	88	8	14	13
jacobi-2d	6	2	14	21	84	10	19	18
lu	5	2	10	12	60	12	23	22
matmul	3	1	3	9	18	11	21	21
mvt	4	2	11	31	68	7	13	12
seidel	3	1	27	37	162	13	24	23
ssymm	8	3	15	33	126	9	19	19

Execution Time Comparison

		Perf. Comparison (Seconds)			
Benchmark	Orig	Par Cur	Par New	Tiled Cur.	Tiled New
gemver	0.31	0.15	0.15	0.15	0.15
mvt	1.40	0.27	0.28	0.42	0.43
seidel	11.8	3.6	3.6	11.5	11.5

More Future Work

Algorithmic:

- Direct UTVPI-UA algorithm similar to LP method
- Integration with Bellman-Ford
- Cost function approximation
- Closeness of UA wrt. original
- Parametrization?

Implementational:

- Per-dependence method integration and feasibility stats
- Duplicate constraints elimination
- Closer integration with PLuTo: YY cases
- Offline/Online method