A Polyhedral Compilation Library with Explicit Disequality Constraints

Sven Verdoolaege

Cerebras Systems

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Outline

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A Motivating Example (Kulkarni and Kruse June 2022)

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for (int i = 0; i < n; i+=1) {
  if (i == p0)
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  if (i == p1)
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  // ...
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$\{\, \mathrm{Stmt}[i] : 0 \leq i < \mathrm{n} \wedge (i < \mathrm{p0} \vee i > \mathrm{p0}) \wedge (i < \mathrm{p1} \vee i > \mathrm{p1}) \wedge (i < \mathrm{p2} \vee i > \mathrm{p2}) \wedge \ldots \}$

$\Rightarrow$ expansion causes explosion in representation

Another Motivating Example (Klebanov 2015)

card { [r1]; [r2]; [r3]; [r4]; [r5]; [r10] }

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$$\text{card} \, \{ \, [\text{r1}]; [\text{r2}]; [\text{r3}]; [\text{r4}]; [\text{r5}]; [\text{r10}] \, \}$$

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$\Rightarrow$ large representation even with explicit disequality constraints

$\Rightarrow$ a lot worse without

Core Representation of Polyhedral Compilation Library

Conjunction of affine inequality constraints

$$\{\, \mathbf{z} : A\mathbf{z} + \mathbf{a} \geq \mathbf{0} \qquad\qquad \}$$

$+$ unions of such sets

No explicit representation for disequality constraints

This applies to libraries

- not supporting existentially quantified variables:
  - `PolyLib` (Wilde 1993)
  - `PPL` (Bagnara et al. 2008)
- supporting existentially quantified variables:
  - `Omega` (Kelly et al. Nov. 1996)
  - `isl` (V. 2010)
  - `Omega+` (Chen June 2012)
  - `FPL` (Pitchanathan et al. Oct. 2021)

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- do not change expressivity
- n equality constraints replace $n + 1$ to $2n$ inequality constraints
- every (independent) equality constraint reduces effective dimensionality

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Why not disequality constraints?

- do not change expressivity
- n disequality constraints avoid split into 2 to 2^n disjuncts

Outline

Seater and Wonnacott (2005)

Detect “inert” disequality constraints

$\Rightarrow$ disequality constraints that can be ignored (in terms of emptiness)

$\Rightarrow$ disequality constraints that involve unbounded direction

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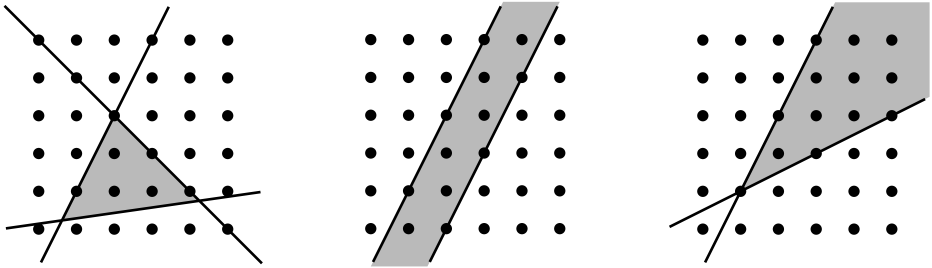

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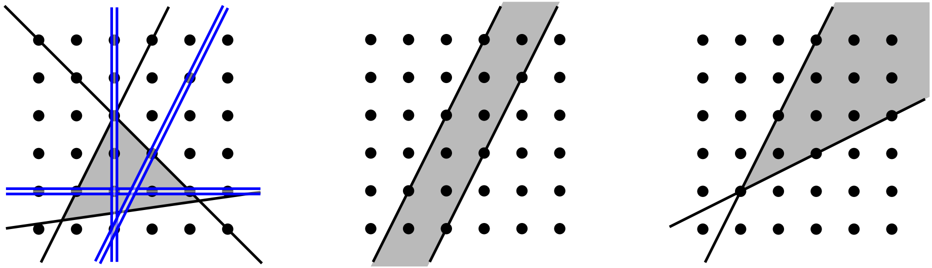

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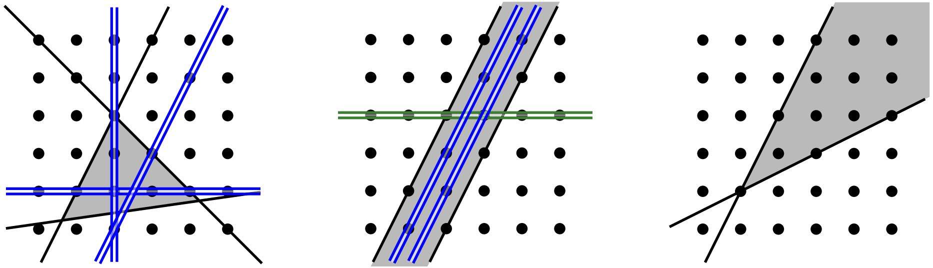

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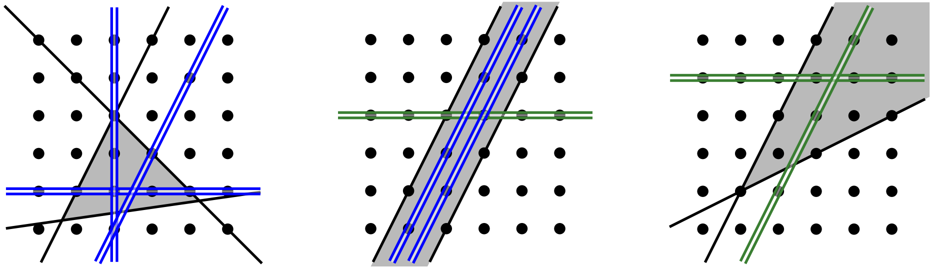

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An equivalent approach would be to simply allow negated equality constraints in simplified relations. This approach could be taken even further, to allow more general negated constraints, or other formulas that cannot be handled efficiently

We do not currently have an implementation of our algorithms, and thus we do not have empirical verification that they are either fast or effective in practice. Given the nature of the changes discussed in the previous section, we do not expect to have an implementation any time soon.

Kulkarni and Kruse (June 2022)

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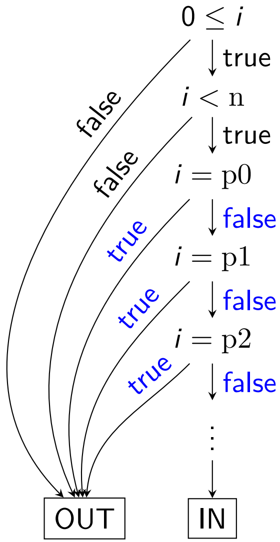

Polyhedral binary decision diagram, PBDD

- internal nodes: affine (in)equality constraints
- terminal nodes: IN: in set; OUT: not in set

⇒ allows negation of (conjunction of) affine constraints (disequality constraint is special case)

However

- limited number of supported operations (intersection, union, subtraction, complement)
- revert to `isl` (with expansion) for other operations
- no support for existentially quantified variables

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Main changes:

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- resolve hidden assumptions
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  For example: `pet` test case

  ```
  { S_5[i=0:99] -> T[i] : i != 57 } % { S_5[i=0:99] : i != 57 };
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  ⇒ `{ S_5[i] -> T[i] : i >= 58 or i <= 56 };` now: `{ S_5[i] -> T[i] }`
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  ```
  for (int c0 = 1; c0 <= 9; c0 += 1) {
    if (c0 != 5) {
      for (int c1 = 1; c1 <= 9; c1 += 1)
        s0(c0, c1);
    } else {
  ```

Extend Internal Representation

Basic set:

$$\{\, \mathbf{z} : A\mathbf{z} + \mathbf{a} \geq \mathbf{0} \wedge B\mathbf{z} + \mathbf{b} = \mathbf{0} \qquad\qquad \}$$

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- $m\, f(\mathbf{z}) + c \neq 0$

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- $c \neq 0$
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- $f(\mathbf{z}) + c \neq 0$
  $f(\mathbf{z}) + a \geq 0$
  - $\Rightarrow$ replace by $f(\mathbf{z}) + a - 1 \geq 0$ if $a = c$
  - $\Rightarrow$ drop disequality if $a < c$

Resolve Hidden Assumptions

Main hidden assumption in `isl`: basic set is convex

Implications:

- all integer values between min/max rational values are in basic set
- simple hull operation can convert 1-disjunct set into basic set
  - $\Rightarrow$ introduce special operation for conversion
  - $\Rightarrow$ simple hull operation drops disequality constraints
  - $\Rightarrow$ another operation for shared constraints needed?

Disequality Constraints in Tableau

Introduce a non-zero variable for each disequality constraint

- a non-zero variable does not participate in pivoting
  - ⇒ always a row variable
- if non-zero variable can attain only negative or only positive values
  - ⇒ non-zero variable is redundant and can be removed
- if non-zero variable can obviously only attain zero value
  - zero values for all remaining columns
  - ⇒ tableau is empty
- sample point only valid if all non-zero variables have non-zero value

$$f_1 = x + y \geq 0$$
$$f_2 = x - 10 \geq 0$$
$$f_3 = y - 5 \geq 0$$
$$f_4 = -y + 5 \geq 0$$
$$f_5 = y - 5 \neq 0$$

		x	y
f_1	0	1	1
f_2	-10	1	0
f_3	-5	0	1
f_4	5	0	-1
f_5	-5	0	1

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Procedure

- trivial solution for 0D and 1D sets
- isolate bounded directions
  - compute recession cone (replace constant terms by 0)
  - (implicit) equality constraints determine bounded directions
  - perform unimodular transformation
- perform backtracking search in tableau on bounded dimensions (can fail)
- pick some corresponding value for unbounded dimensions (always succeeds)

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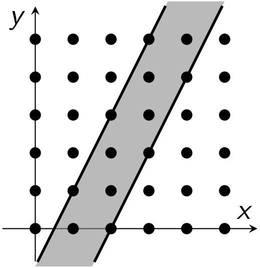

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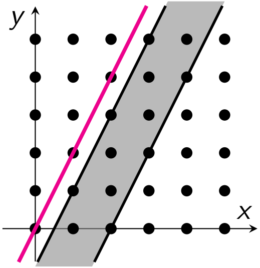

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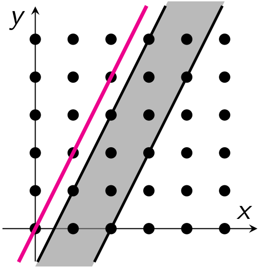

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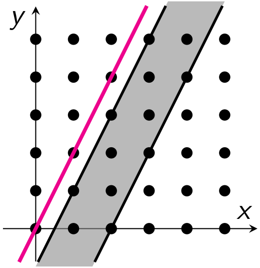

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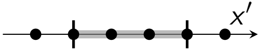

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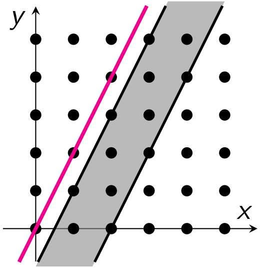

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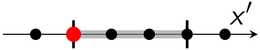

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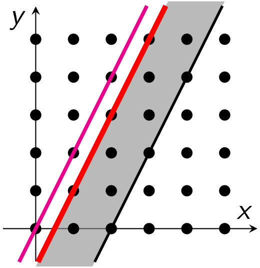

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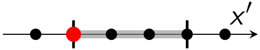

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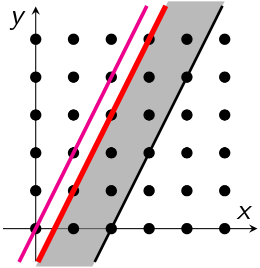

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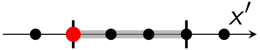

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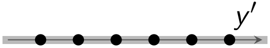

Picking Element in Unbounded Set

Rational element can easily be picked in tableau (sample value, possibly non-integer values)

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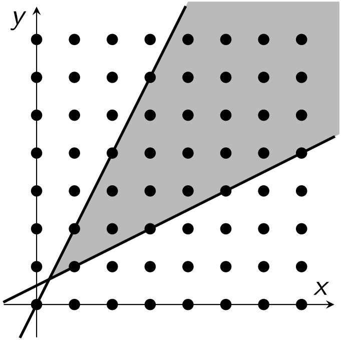

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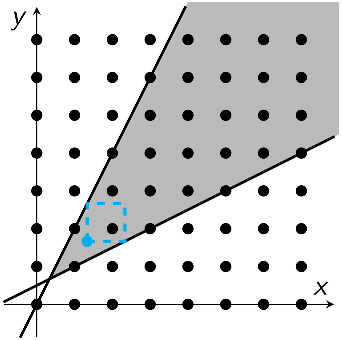

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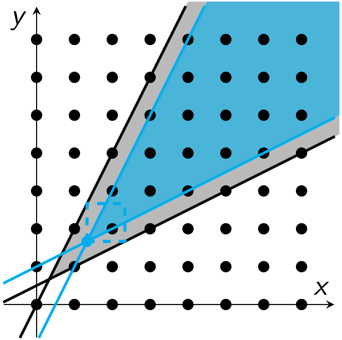


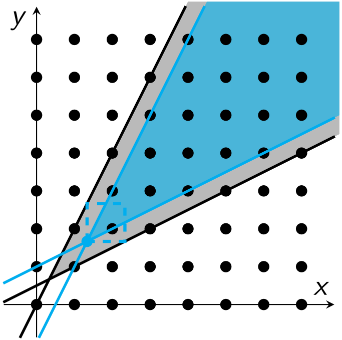
$$\{ [x, y] : y \leq 2x \wedge x \leq 2y - 1 \}$$

$$\{ [x, y] : y \leq 2x - 1 \wedge x \leq 2y - 2 \}$$

Picking Element in Unbounded Set

Rational element can easily be picked in tableau (sample value, possibly non-integer values)

- $\Rightarrow$ restrict set to points that have entire unit cube included in original set
- $\Rightarrow$ pick rational element in restricted set $(4/3, 5/3)$
- $\Rightarrow$ round up



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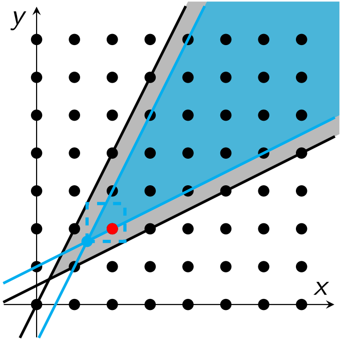

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Emptiness and Sampling

- Sampling picks an integer element
- Set is empty if it has no integer elements

Procedure

- trivial solution for 0D and 1D sets
- isolate bounded directions
  - compute recession cone (replace constant terms by 0)
  - (implicit) equality constraints determine bounded directions
  - perform unimodular transformation
- perform backtracking search in tableau on bounded dimensions (can fail)
- pick some corresponding value for unbounded dimensions (always succeeds)

Emptiness and Sampling

- Sampling picks an integer element
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Procedure

- trivial solution for 0D and 1D sets
- isolate bounded directions
  - compute recession cone (replace constant terms by 0) ignoring disequality constraints
  - (implicit) equality constraints determine bounded directions
  - perform unimodular transformation
- perform backtracking search in tableau on bounded dimensions (can fail)
  - drop disequality constraints involving unbounded dimension ("inert")
  - skip values violating any other disequality constraint
- pick some corresponding value for unbounded dimensions (always succeeds)

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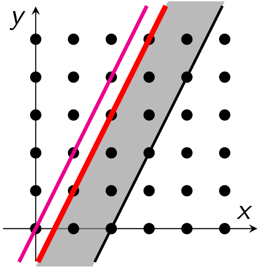
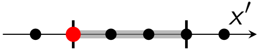


$$\{\, [x, y] : 1 \le 2x - y \le 4 \,\}$$

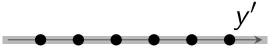
$$\{\, [x, y] : 0 \le 2x - y \le 0 \,\}$$

$$x' = 2x - y$$

$$y' = y$$



$$\{\, [x'] : 1 \le x' \le 4 \,\}$$



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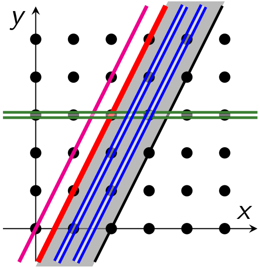

$\{ [x, y] : 1 \leq 2x - y \leq 4 \}$

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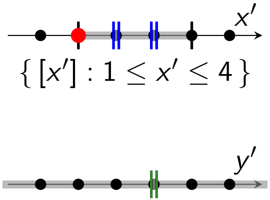

$\{ [x'] : 1 \leq x' \leq 4 \}$

Picking Element in Unbounded Set

Rational element can easily be picked in tableau (sample value, possibly non-integer values)

$\Rightarrow$ restrict set to points that have entire unit cube included in original set

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Picking Element in Unbounded Set

Rational element can easily be picked in tableau (sample value, possibly non-integer values)

- ⇒ restrict set to points that have entire $(1 + n^{\neq})$-cube included in original set
- ⇒ pick rational element in restricted set
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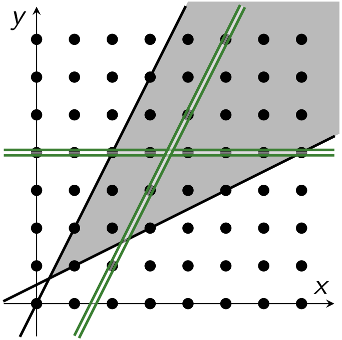

$$\{ [x, y] : y \leq 2x \land x \leq 2y - 1 \}$$

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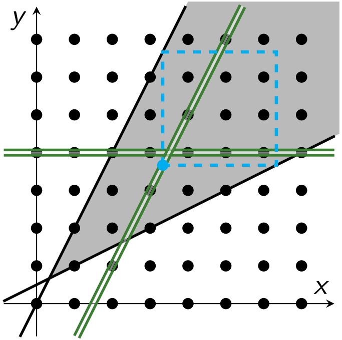

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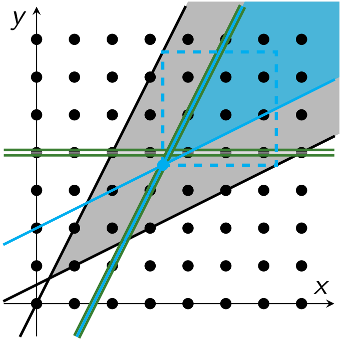

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Picking Element in Unbounded Set

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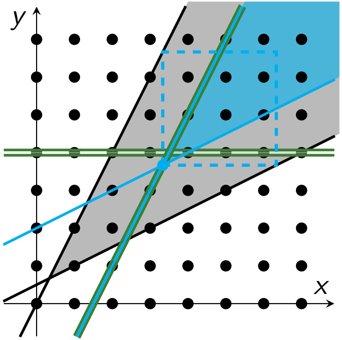

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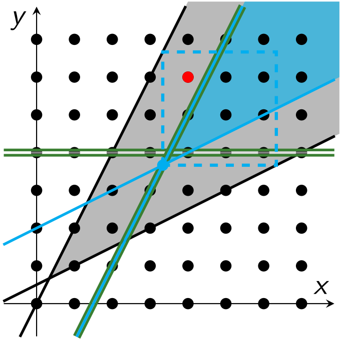
- $\Rightarrow$ round up (skipping violated disequality constraints) $(4, 6)$



$$\{ [x, y] : y \leq 2x \wedge x \leq 2y - 1 \}$$

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Redundant Local Variables

Basic set:

$$\{\, \mathbf{x} : \exists \boldsymbol{\alpha} : A_1 \mathbf{x} + A_2 \boldsymbol{\alpha} + \mathbf{a} \geq \mathbf{0} \,\}$$

Some local variable $\boldsymbol{\alpha}$ may be redundant

Some of these can be detected based purely on constraints

- consider all pairs of lower and upper bounds on variable α
- if each pair admits an integer value
  $\Rightarrow$ α can be eliminated (using Fourier-Motzkin)

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- α is involved in n disequality constraints
- if each pair admits $1 + n$ integer values
  - $\Rightarrow$ α can be eliminated (using Fourier-Motzkin)

$\Rightarrow$ potential trade-off between number of disjuncts and dimensionality of disjuncts

Parametric Integer Programming

Compute lexicographic minimum of some variables **x** in terms of other variables **n**

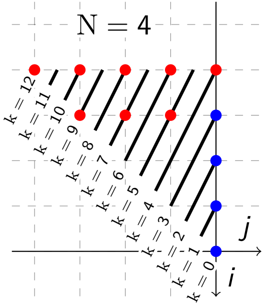
Two tableaux:

- main tableau in **x** and **n**
- context tableau **n**

Pivoting in main tableau depends on sign of symbolic constant term in context tableau

$\Rightarrow$ requires context splits if constant term can attain both positive and negative values

$$R = \{ [i,j] : 0 \le -i \le \mathrm{N} \wedge 0 \le -j \le -i \wedge 0 \le \mathrm{k} \le 3\mathrm{N} \wedge \mathrm{k} = -i - 2j \}$$



lexmin R =

if $\mathrm{k} < \mathrm{N}$

$[-\mathrm{k}, 0]$

else

if $3\left\lfloor \frac{\mathrm{k}+\mathrm{N}}{2} \right\rfloor \ge 2\mathrm{k}$

$\left[\mathrm{k} - 2\left\lfloor \frac{\mathrm{k}+\mathrm{N}}{2} \right\rfloor, -\mathrm{k} + \left\lfloor \frac{\mathrm{k}+\mathrm{N}}{2} \right\rfloor\right]$

Parametric Integer Programming

Compute lexicographic minimum of some variables $\mathbf{x}$ in terms of other variables $\mathbf{n}$

Two tableaux:

- main tableau in $\mathbf{x}$ and $\mathbf{n}$
- context tableau $\mathbf{n}$

Pivoting in main tableau depends on sign of symbolic constant term in context tableau

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Keep track of disequality constraints in tableaux

If disequality constraint $g(\mathbf{n}, \mathbf{x}) \neq 0$ may be violated by potential solution

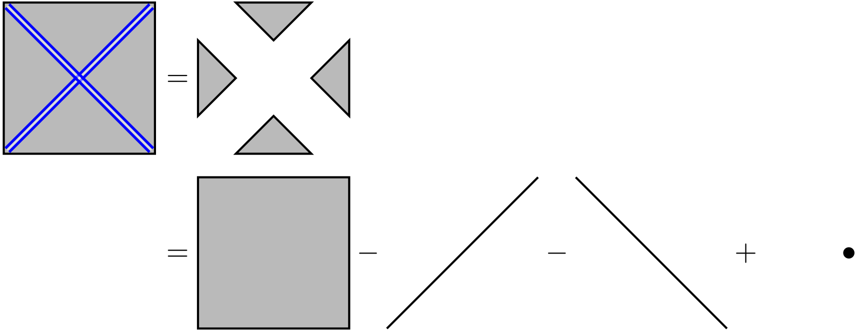
$\Rightarrow$ split context into 2 cases

- $f(\mathbf{n}) \neq 0$ (implying $g(\mathbf{n}, \mathbf{x}) \neq 0$ is not violated)
  $\Rightarrow$ proceed with other disequality constraints
- $f(\mathbf{n}) = 0$ (implying $g(\mathbf{n}, \mathbf{x}) \neq 0$ is violated)
  - $\Rightarrow$ compute two solutions, for $g(\mathbf{n}, \mathbf{x}) \geq 1$ and $g(\mathbf{n}, \mathbf{x}) \leq -1$
  - $\Rightarrow$ take minimum of two solutions

Splitting $g(\mathbf{n}, \mathbf{x}) \neq 0$ up front computes same minimum but then cost is always incurred

Some Other Operations

- Preparation for counting using `barvinok` (V., Seghir, et al. June 2007)



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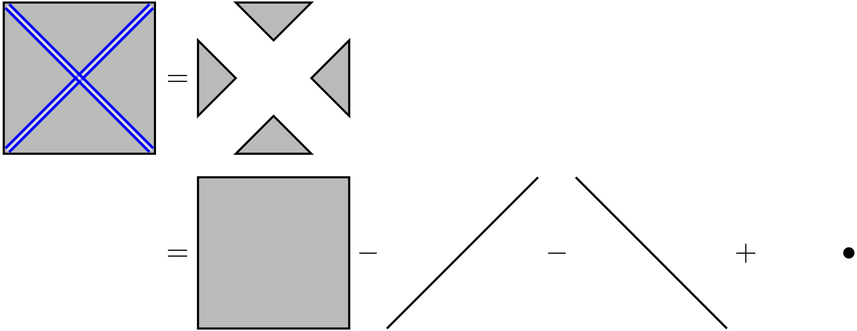

- Transitive closure approximation
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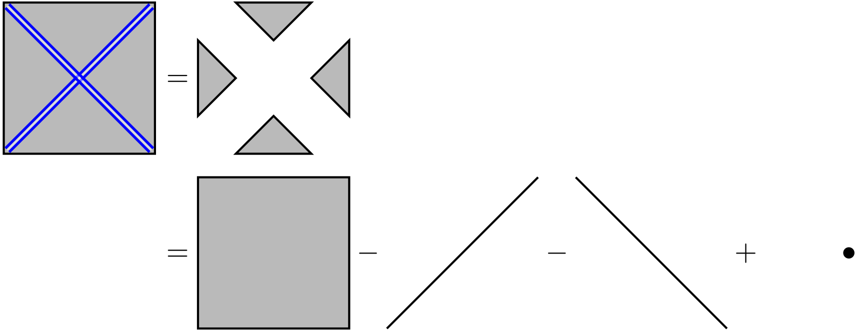
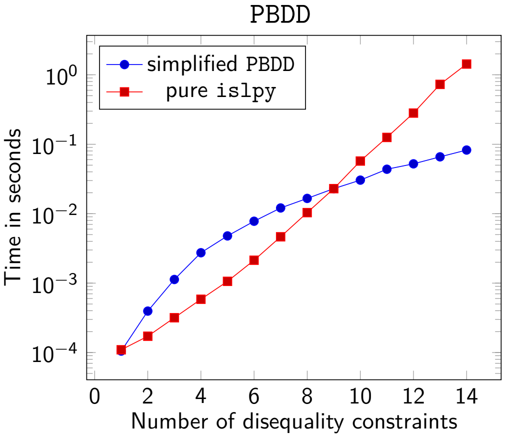


- Transitive closure approximation
  Basic sets do not have to be split but result may be less accurate
- Scheduling
  Disequality constraints essentially ignored

Outline

PBDD versus isl with Explicit Disequality Constraints

$$\{\, [i] : \bigwedge_{j \leq n^{\neq}} i \neq \mathrm{p}_j \,\}$$



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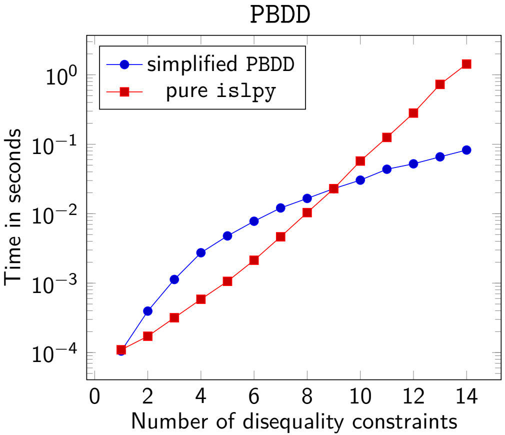
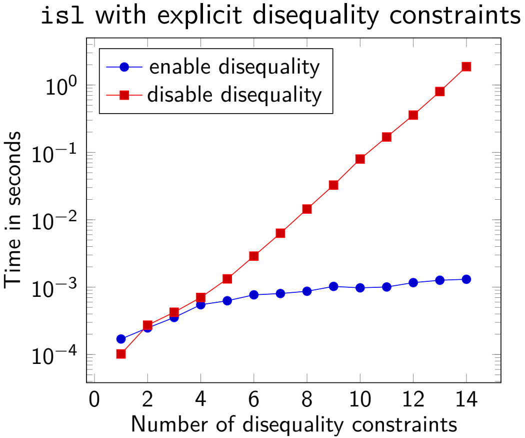




Note: construction times with PBDD and isl not directly comparable

Full Polyhedral Compilation Flow

```
for (int i = 0; i < n; ++i) {
  if (i == p0 || i == p1 || i == p2)
    continue;
  A[i] = i;
}
for (int i = 0; i < n; ++i) {
  if (i == p0 || i == p1 || i == p2)
    continue;
  B[i] = A[i];
}
```

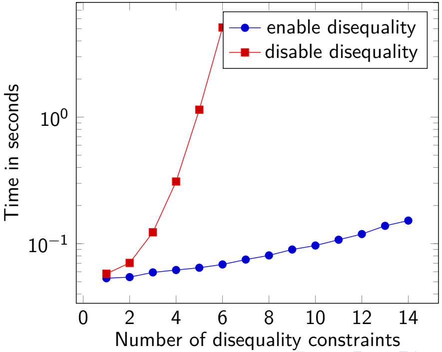
PPCG (V., Juega, et al. 2013) output:

```
for (int c0 = 0; c0 < n; c0 += 1)
  if (c0 != p0 && c0 != p1 && c0 != p2) {
    A[c0] = (c0);
    B[c0] = A[c0];
  }
```

(No changes required to PPCG)

Involves

- construction of polyhedral model
- dependence analysis
- scheduling
- AST generation



Outline

Conclusion

Supporting explicit disequality constraints in a polyhedral compilation library is feasible

- requires only conceptually minor adjustments
- in some cases simply delaying split to where it becomes relevant
- can dramatically reduce size of representation
- mostly transparent to the user

Some trade-offs involved, e.g.,

- elimination of redundant local variables
- accuracy of transitive closure approximation

Perhaps useful to consider other explicit constraints, e.g.,

- lexicographic constraints

Outline

Incremental LP Solver

Core representation: tableau

Given

$$\{\, \mathbf{z} : A\mathbf{z} + \mathbf{a} \geq \mathbf{0} \,\}$$

with n variables $\mathbf{z}$ and m constraints $A\mathbf{z} + \mathbf{a} \geq \mathbf{0}$

- introduce a non-negative variable f_i for each affine expression
- tableau writes m variables in terms of n variables
- initially, $\mathbf{f}$ in terms of $\mathbf{z}$

$$f_1 = x + y \geq 0$$
$$f_2 = x - 10 \geq 0$$
$$f_3 = y - 5 \geq 0$$
$$f_4 = -y + 5 \geq 0$$

		x	y
f_1	0	1	1
f_2	-10	1	0
f_3	-5	0	1
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Incremental LP Solver

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- sample value: assign zero to all column variables
  $x = 0, y = 0, f_1 = 0, f_2 = -10, f_3 = -5, f_4 = 5$

Incremental LP Solver

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- a pivot interchanges a row and a column variable

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  $f_3 = -5 + y \Rightarrow y = 5 + f_3,\ f_1 = 5 + x + f_3,\ f_4 = -f_3$

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  - $\Rightarrow$ no longer participates in pivoting

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- a column variable that is known to be zero (e.g., f_3) can be "killed"
  - $\Rightarrow$ fixed to zero
  - $\Rightarrow$ no longer participates in pivoting
- sample value is "valid" if
  - all non-negative variables have non-negative value
  - all variables have integer value

$\Rightarrow$ current sample value not valid because $f_2 = -10$

Disequality Constraints in Tableau

Introduce a non-zero variable for each disequality constraint

- a non-zero variable does not participate in pivoting
  - ⇒ always a row variable
- if non-zero variable can attain only negative or only positive values (while non-negative variables have non-negative values)
  - ⇒ non-zero variable is redundant and can be removed
- if non-zero variable can obviously only attain zero value
  - zero sample value
  - all coefficients in non-killed columns are zero
  - ⇒ tableau is empty

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f_3	-5	0	1
f_4	5	0	-1

		x	f_3
f_1	5	1	1
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y	5	0	1
f_4	0	0	-1

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Disequality Constraints in Tableau

Introduce a non-zero variable for each disequality constraint

- a non-zero variable does not participate in pivoting
  - ⇒ always a row variable
- if non-zero variable can attain only negative or only positive values (while non-negative variables have non-negative values)
  - ⇒ non-zero variable is redundant and can be removed
- if non-zero variable can obviously only attain zero value
  - ▸ zero sample value
  - ▸ all coefficients in non-killed columns are zero
  - ⇒ tableau is empty

$$f_1 = x + y \geq 0$$
$$f_2 = x - 10 \geq 0$$
$$f_3 = y - 5 \geq 0$$
$$f_4 = -y + 5 \geq 0$$
$$f_5 = y - 5 \neq 0$$

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Effect on sample value validity

- sample value is "valid" if
  - all non-zero variables have non-zero value
  - all non-negative variables have non-negative value
  - all variables have integer value

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