Reuse Analysis via Affine Factorization

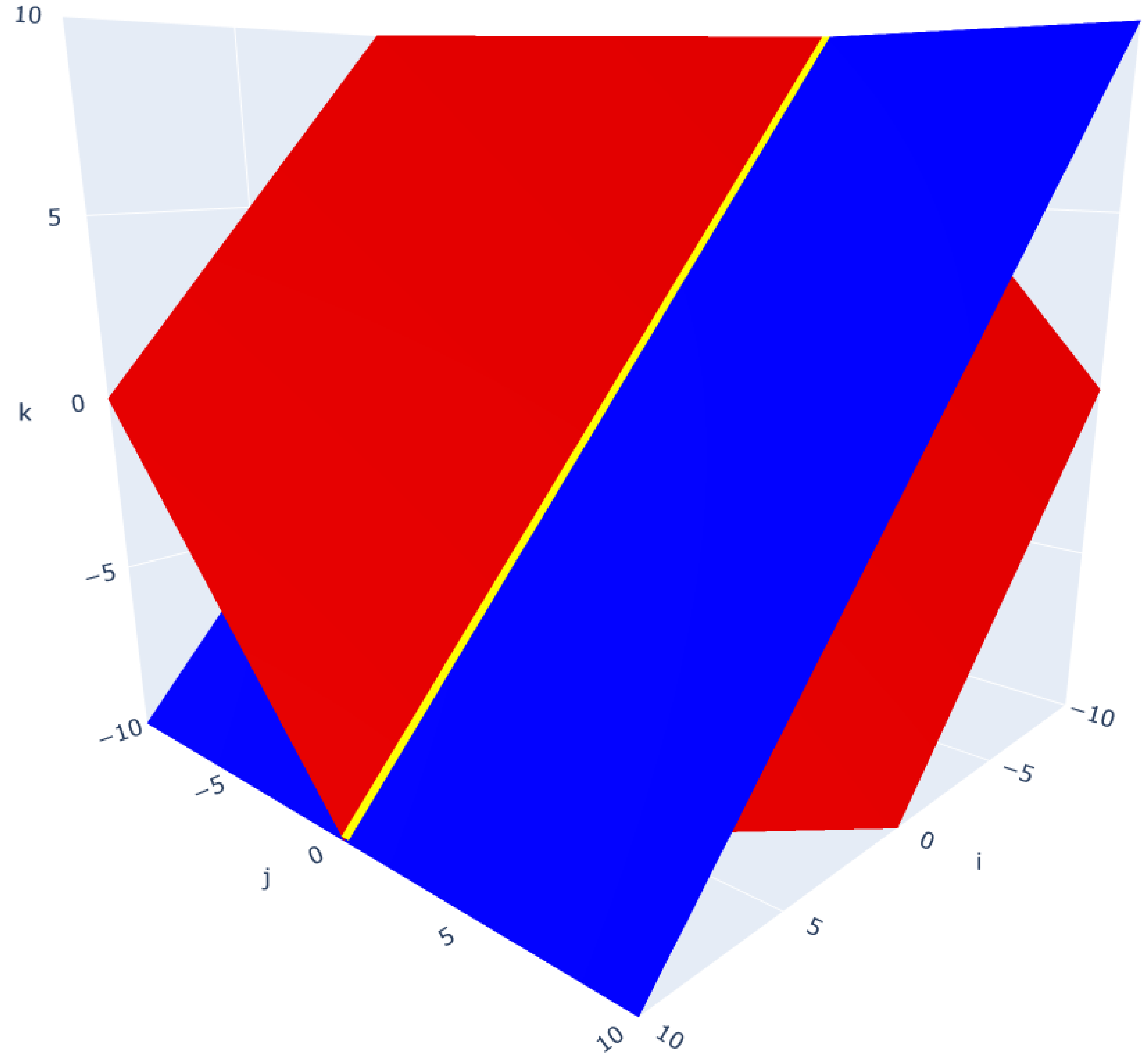
Ryan Job

Advisor: Prof. Sanjay Rajopadhye

Motivation

- Consider the 3D expression here.
- Each plane reads the same value from an input.
  - The blue plane reads the same value of A
  - The red plane reads the same value of B.
- Their intersection produces the same results in Y.
- We have a 2D computation in 3D space.
- How do we automatically detect and exploit this?

$$Y[i,j,k] = A[i+k] + B[i+j+k]$$



Outline

- Background
- Affine Factorization Algorithm
- Automating Reduction Simplification

Background

Affine Maps and Matrix Notation

- Affine maps apply a linear transformation and a translation to a domain.
  - $y = Ax + b$
- We use an augmented matrix notation:
  - Augment the input (x) with a constant 1.
  - Merge the transformation (A) with the translation (b).
  - $y = [A|b] \cdot \begin{bmatrix} x \\ 1 \end{bmatrix}$

Hermite Normal Form (HNF)

- HNF is analogous to reduced row echelon form (RREF), but for integer spaces.
- Given a set of vectors, HNF finds a basis which spans them.
- We use HNF to decompose a matrix into two:
  - M: input vectors, written as rows of a matrix.
  - H: the basis of the input vectors.
  - U^{-1}: a transformation from the basis to the input.
  - $M = U^{-1} \cdot H$

Affine Factorization Algorithm

Algorithm Overview

- We use HNF to “factorize” a set of affine maps with a common domain.
- Find the smallest subspace of distinct values for the computation.
  - We call this the “intermediate space”.
- Rewrite the original maps as the composition of two:
  - H maps the domain to the intermediate space.
  - Then, subsets of U^{-1} map to the desired ranges.
- This use case is mathematically simple, but we could not find it in use.
  - Neither in the polyhedral community nor the wider compilation community.

Algorithm Details

- Write the affine maps as augmented matrices.
- Concatenate the matrices on top of each other: M.
- HNF is used to rewrite the maps with H and U^{-1}.
  - Each map uses H as-is, and a subset of U^{-1}.
- Since H is common to all rewrites, it can be factored out.
  - Introduce a new variable of only the unique values (U^{-1}).
  - Map the full output to these values (H).

Algorithm 1 Algorithm for factorizing affine maps

Input: A list matrices M_i representing affine maps, all with the same D-dimensional domain.
Output: A common right factor H and left factors Q_i.

procedure FACTORIZEMAPS($M_0 \dots M_n$)
  $M \leftarrow$ CONCATENATE($M_0 \dots M_n$)
  $H, U \leftarrow$ HERMITENORMALFORM(M)
  $Q \leftarrow$ MATRIXINVERSE(U)
  for r = ROWS(H) $-1 \dots 0$ **do**
    if ISROWOFZEROS(H, r) **then**
      $H \leftarrow$ DROPROW(H, r)
      $Q \leftarrow$ DROPCOL(Q, r)
    end if
  end for
  $start \leftarrow 0$
  for $i = 0 \dots n$ **do**
    $end \leftarrow start +$ ROWS(M_i)
    $Q_i \leftarrow$ GETROWS($Q, start, end$)
    $start \leftarrow end$
  end for
  return $H, Q_0 \dots Q_n$
end procedure

Automating Reduction Simplification

Alpha & AlphaZ

- Alpha is a declarative, equational language for the polyhedral model.
- Reductions are modeled as a collection of inputs combined with an operator.
- AlphaZ is a system for optimizing Alpha equations and generating C code.
- We are focusing on the “Simplifying Reductions” optimization.
  - Exploits reused values to lower the asymptotic time complexity of the computation.
- Currently, it requires human input to indicate how values are reused.
  - Given this information, the reduction can be automatically rewritten.

Automatic Reduction Simplification

- We apply affine factorization to the affine maps which index input variables.
- If the space of unique values is lower dimension than the result:
  - Values are reused throughout the computation.
  - The basis, H, will have a non-trivial null space.
- Vectors in this null space indicate how values are reused.
- Any such vector is enough information to automate Simplifying Reductions.

Current Status

- Developed a proof-of-concept for affine factorization.
  - Publicly available on GitHub (link in the paper).
  - Presented as a Jupyter notebook using the islpy library.
- Incorporating the algorithm into AlphaZ.
  - Goal: automate the Simplifying Reductions optimization.

Additional Uses

- Found a use case for memory layout transformations in FPGA accelerators.
  - Relates to work by Corentin Ferry, being presented later today.
- Investigating applications to algorithm-based fault tolerance.
  - Relates to work by Louis Narmour, presented at IMPACT last year.
- We hope to hear from you about more use cases!

Thank you!